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LETTER TO THE EDITOR

A regular-random fractal model for cluster numbers and structure in percolation

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Abstract. A regular-random fractal, intermediate between statistical fractals and deterministic fractals, is presented to imitate the geometric texture at and near percolation threshold. The model is constructed on the square lattice via a rule of bond occupation with use of position space renormalisation group. It shows the typical percolation behaviour as a function of a parameter p (the bond concentration). The critical bond concentration, correlation length exponent and scaling property of the cluster size distribution are found. The scaling relations between critical exponents for cluster numbers and structure are shown to be exactly satisfied. The fractal dimension of its cutting bonds agrees with the inverse of the connectedness length exponent at criticality.

The geometry and statistics of clusters is one of the most important problems in phase transitions and critical phenomena. The properties of cluster numbers and structure in percolation have been reviewed by Stauffer (1979, 1985). Fractals and the concept of fractal dimensionality have recently received considerable attention owing to their applicability to the percolating infinite cluster (Mandelbrot 1982, Given and Mandelbrot 1983, Ben-Avraham and Havlin 1983, Stanley and Coniglio 1983, Kapitulnik and Deutscher 1984, Stanley 1985, 1986). Currently, fractals fall into three completely distinct classes: deterministic fractals (such as the Sierpinski gasket), statistical fractals (such as percolation clusters) and regular-random fractals (Martin and Keefer 1985) (intermediate between deterministic and statistical fractals). Fractal models have been proposed to imitate the infinite cluster and its backbone at the percolation threshold, i.e. the family of Sierpinski gaskets (Gefen *et al* 1981), the Mandelbrot-Koch curves (Mandelbrot 1984a, b, Mandelbrot and Given 1984) and the regular-random fractals (Kirkpatrick 1979, Martin and Keefer 1985). These models possess geometric and topological properties very close to the infinite cluster at threshold but do not describe the approaches towards the threshold. Nagatani (1985, 1986) proposed the deterministic fractal model to imitate the geometric texture near the threshold. It was found that the typical percolation behaviour was successfully reproduced as a function of bond concentration p near the threshold. The critical concentration, correlation length exponent and scaling property of the cluster size distribution were found. These models are not satisfied by Coniglio's relation (1982): $D_c = 1/\nu$ where D_c and ν indicate the fractal dimension of cutting bonds and the connectedness length exponent, respectively.

In this letter, we propose a regular-random fractal model to imitate the geometric textures of cluster numbers and structure at and near the percolation threshold. We shall show that the model is satisfied by all the conventional scaling relations between critical exponents for cluster numbers and structure and Coniglio's relation. The model

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also presents a fractal geometric picture for the position space renormalisation group. We restrict ourselves to the bond percolation problem on the square lattice. Consider a square superlattice made by nodes separated by a distance of $\xi = b^N$, connected by quasilinear links with a hierarchical structure. The lattice model is self-similar on smaller length scales than the connectedness length ξ, but becomes a homogeneous square lattice on large length scales. Three construction stages of the lattice model are shown in figure 1. Each bond on the square lattice is replaced by a cell with four bonds of length $\frac{1}{2}$ (figure 1(b)). The dotted lines represent electrical connections with the original nodes. Each bond on the resultant lattice is furthermore replaced by a cell (figure 1(c)). The process is continued to the N stages. We note that a position space renormalisation is exactly applicable to the resultant lattice model because of a hierarchical structure. Each bond on the lattice after the Nth stages is occupied with probability p_N and vacant with probability $1-p_N$. The recursion relation for p_N, the occupation probability, satisfies (Hong 1984)

$$p_N = R(p_{N-1}) = 2p_{N-1}^2 - p_{N-1}^4 \tag{1}$$

which has the fixed point $p^* = (\sqrt{5}-1)/2 = 0.618$.

Figure 2 shows spanning and non-spanning configurations for the $b = 2$ cell. Spanning clusters are indicated by figures 2(a), (b) and (c). Non-spanning clusters are represented by figures 2(d), (e) and (f). We define that the cluster of two bonds in configuration (e) is constructed by two unconnected bonds. We consider a stepwise generation of the regular-random lattice model, to correspond with the regular-random fractal designed by Martin and Keefer (1985). In figure 1(a), a bond is occupied with probability p_0 and unoccupied with $1-p_0$. If the bond is present, the bond is replaced with a spanning cluster in figures 2(a), (b) and (c), and otherwise with a non-spanning cluster in figures 2(d), (e) and (f), where the occupation probability p_1 is given by $p_1 = R^{-1}(p_0)$. Furthermore, the second-order generation is obtained by replacing each occupied bond with a spanning cluster and each unoccupied bond with a non-spanning cluster, where $p_2 = R^{-1}(p_1)$. The process is continued to the Nth-order generation (see figure 3). In the limit where N is sufficiently large, the resultant lattice approaches the percolation threshold.

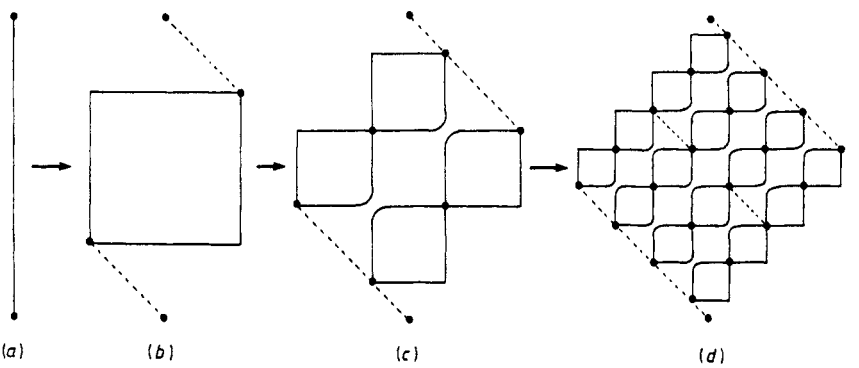

Figure 1. Three construction stages of a square superlattice with a hierarchical structure. Each bond (a) on the square lattice is replaced by a cell with four bonds of the length $1/b$ ($b = 2$) (b). The dotted lines represent electrical connections with the original nodes. Each bond on the resultant lattice (b) is furthermore replaced by a cell (c). The process is continued to the N stages.

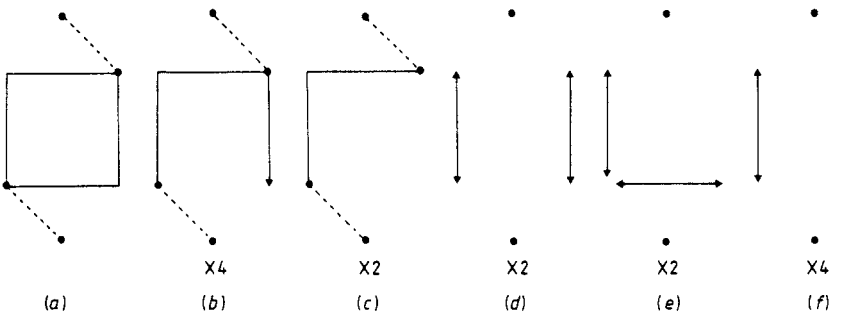

Figure 2. Spanning and non-spanning configurations for the $b = 2$ cell. Spanning clusters are shown in (a), (b) and (c) are connected by the electrical connections indicated by dotted lines at the original nodes. Non-spanning clusters are represented by (d), (e) and (f). The cluster of two bonds in the configuration (e) is defined to be constructed by two unconnected bonds. The non-spanning clusters generate finite clusters in percolation. Unconnected vertices of bonds are marked by triangles. The number below each figure indicates that of the same configuration.

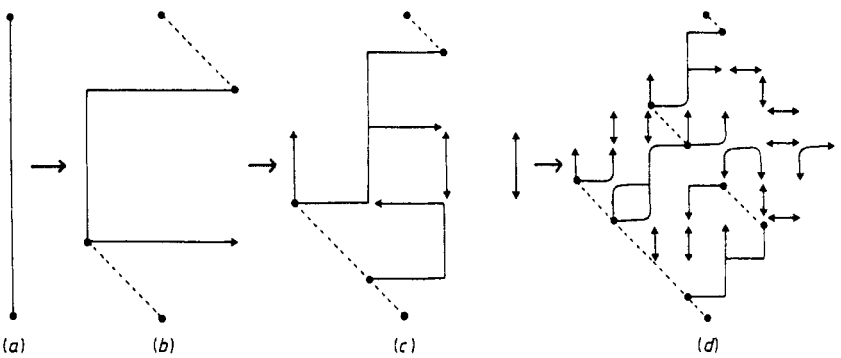

Figure 3. The first three generations of the regular-random model. A bond is occupied with probability p_0 at the zeroth generation (a). At the first generation (b) the bond is replaced with a spanning cluster in figures $2(a)$, (b) and (c) where the occupation probability is given by $p_1 = R^{-1}(p_0)$. At the second generation (c) each occupied bond and unoccupied bond are, respectively, replaced with a spanning cluster and an unspanning cluster. The process is continued to the Nth generation. Unoccupied bonds generated finite clusters successively. Unconnected vertices are marked by triangles.

When one has the initial value $p_0 = p^*$, the resultant lattice represents the geometric texture at the threshold. On the other hand, if $p_0 = 1 - \varepsilon$ or $p_0 = \varepsilon$ $(0 < \varepsilon \ll 1)$, then the lattice represents the geometric texture above or below the threshold. We note that the geometric texture at $p_0 = p^*$ gives the regular-random fractal exactly, but the geometric texture at $p_0 = 1 - \varepsilon$ (or $p_0 = \varepsilon$) becomes the fractal in the range of length L: $L \ll \xi$ $(\xi = 2^N)$.

The system obtained consists of the islands separated from the percolating network and a superlattice made by nodes separated by a distance $\xi = 2^N$, connected by quasilinear links. In general, every lattice bond has three choices: it can be empty, with probability $1 - p$ $(p = p_N)$; it can be part of the infinite network of occupied bonds with probability pP_∞ (where P_∞ is the percolation probability), or it can be part of one of the many finite clusters, with probability $p(1 - P_\infty)$. Since each s cluster contains exactly s bonds, the probability of any lattice bond belonging to an s cluster is $P_s = sn_s$

(n_s is the number of s clusters divided by the total number of lattice bonds). The sum of all these probabilities equals unity. As the concentration p approaches the threshold p_c ($=p^*$), the pair connectedness length ξ diverges, $\xi \sim (p-p_c)^{-\nu}$. The exponent of the connectedness length is given by

$$\nu = \ln 2/\ln(\mathrm{d}R/\mathrm{d}p)_{p^*} = 1.635. \tag{2}$$

At the threshold, the fractal dimension D of the infinite cluster is given by

$$\begin{aligned} D &= \ln\langle\langle n_i\rangle\rangle^*/\ln b \\ &= 1.611 \end{aligned} \tag{3}$$

where the asterisk indicates the value at the fixed point and $\langle\langle n_i\rangle\rangle$ is the average number of bonds within the spanning cluster if the cell is connected:

$$\begin{aligned} \langle\langle n_i\rangle\rangle &= \langle n_i\rangle / R(p) \\ &= (4p^4 + 12p^3q + 4p^2q^2)/R(p). \end{aligned} \tag{4}$$

Similarly, we derive the fractal dimensions of the backbone and its cutting bonds:

$$D_b = \ln\langle\langle n_b\rangle\rangle^*/\ln b \qquad D_c = \ln\langle\langle n_c\rangle\rangle^*/\ln b \tag{5}$$

where n_b and n_c represent, respectively, the number of bonds through which electrical current flows in the spanning cluster and the number of bonds such that if one is cut the entrances are no longer connected to the exits in the cell:

$$\begin{aligned} \langle\langle n_b\rangle\rangle &= (4p^4 + 8p^3q + 4p^2q^2)/R(p) \\ \langle\langle n_c\rangle\rangle &= (8p^3q + 4p^2q^2)/R(p). \end{aligned} \tag{6}$$

We obtain the following relation:

$$\mathrm{d}R/\mathrm{d}p\big|_{p^*} = \langle\langle n_c\rangle\rangle^*. \tag{7}$$

The rigorous relation $D_c = 1/\nu$ derived by Coniglio (1982) holds for this model. The fractal dimension of cutting bonds agrees with the inverse of the connectedness length exponent at criticality. We thus obtain explicit expressions for the quantities characterising cluster structure.

The other important feature of the regular-random model described above is that it is possible to get explicit expressions for the quantities characterising cluster numbers. We define $\langle N_N\rangle$, the average of the total number of bonds at the Nth stage, and $\langle n_N\rangle$, the average number of finite clusters generated at the Nth stage. The following relation is obtained:

$$\langle N_N\rangle = \langle n_N\rangle + \langle\langle n_{i,N}\rangle\rangle\langle N_{N-1}\rangle \tag{8}$$

where $\langle\langle n_{i,N}\rangle\rangle$ represents the average number of bonds within a spanning cluster generated at the Nth stage when the cell is connected. We obtain

$$\langle n_N\rangle/(b^2)^N = \langle n_{f,N}\rangle/b^2 \tag{9}$$

where $\langle n_{f,N}\rangle$ indicates the average number of bonds within the cell of non-spanning configurations (figures 2(d), (e) and (f)) at the Nth stage. In order to obtain the cluster size distribution, one should note that the clusters generated in the kth stage, on average, contain $s(k) \sim (b)^D$ bonds. At the threshold, the cluster size distributions consist of

$$n_s^* \sim \sum_{k=1}^{\infty} (1/b^2)^k \delta(s - (b^D)^k). \tag{10}$$

By spreading the delta functions over the interval we obtain

$$n_s^* \sim s^{-\tau} \qquad \tau = 1 + d/D. \tag{11}$$

Near the threshold, the regular-random model is self-similar (fractal) on smaller length scales than the connectedness length, but becomes homogeneous on large length scales. The largest finite clusters contain $s_c (= \xi^D)$ bonds and finite clusters, larger than s_c, do not exist. One can arrive at the scaling form near the threshold

$$n_s \sim s^{-(1+d/D)} f(|p - p_c|^{1/\sigma} s) \qquad 1/\sigma = \nu D \tag{12}$$

where the function $f(x)$ is zero for $x \geq 1$.

The similar scaling form for the radius of cluster holds:

$$R_s \sim s^{1/D} g(|p - p_c|^{1/\sigma} s). \tag{13}$$

All the scaling relations between critical exponents for cluster numbers and structure (Stauffer 1985) are satisfied by our model:

$$\begin{aligned} &\beta = (\tau - 2)/\sigma \qquad \gamma = (3 - \tau)/\sigma \qquad (2 - \alpha) = (\tau - 1)/\sigma \\ &1/D = \sigma\nu \qquad d\nu = (\tau - 1)/\sigma. \end{aligned} \tag{14}$$

In summary, the geometric texture in percolation can be imitated by the regular-random model with the use of a position space renormalisation group. The model shows the typical percolation behaviour as a function of the bond concentration p. The connectedness length exponent is obtained for characterising the approaches towards the percolation threshold. All the conventional scaling relations between critical exponents for cluster numbers and structure are satisfied in this construction. Conglio's relation for the cutting bonds is also satisfied.

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